REGULARIZED ANALYTICAL SOLUTION OF CAUCHY PROBLEM FOR ELASTIC RECTANGLE

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1. Introduction

Let us have an element of construction with a boundary $S = S_1 \cup S_2$ consisting of two parts S_1 and S_2. The part S_1 is accessible for all measurements and the part S_2 is not accessible. In engineering practice it is often necessary to reconstruct the stress-strain state of such structural element by means of measured data on S_1. In this case within an elastic model we have the Cauchy problem for the Lame equation of theory of elasticity. The Cauchy problem for elliptic equations is a so-called ill-posed problem. The main difficulty for solving such problems is numerical instability. For the Cauchy problem in linear elasticity there exist numerous ways for numerical solving. Many of these methods can be classified as the Tikhonov type regularization methods, the finite element method, the boundary element method, etc. (sf. [1, 2]). With regard to analytical solutions it is known only usage of Carleman's method in the elasticity theory. In this paper we present an analytical method for the Cauchy problem solution of Lame equation in a rectangle. This method generalizes the Liu method [3] for an analytical solution of the Cauchy problem of Laplace equation in the rectangle.

2. Formulation of problem

In numerous publications under the name of Cauchy problem for the Laplace equation there exist two types of problems. In them it is necessary to find a harmonic function in $\Omega \in \mathbb{R}^3$. At first one has a cylindrical domain $\Omega = S \times (0, T)$ with a lower base $S \in \mathbb{R}^2$ and a lateral surface S_1. On S the Cauchy conditions are given: $u = f$, $\partial u \backslash \partial n = g$, and some boundary conditions are given on the lateral surface S_1, for example, $u = 0$, on an upper base there are no given boundary conditions. Such problems can be named as an initial boundary value problem for the Laplace equation (see Ch.9 in [1]). For another type we have a bounded domain $\Omega \in \mathbb{R}^3$ with a smooth boundary $S = S_1 \cup S_2$ and the Cauchy conditions are given on the part S_1 of a boundary S. We will deal with the first type of the Cauchy problem and it's analogous in the elasticity theory. It is necessary to note that the first type of Cauchy problem is not considered for analytical solutions in the elasticity theory. All analytical results in the elasticity theory concern the second type problem and use Carleman's method.

An equilibrium equation in the theory of elasticity is the Lame equation

$$L\mathbf{u} \equiv \mu \triangle \mathbf{u} + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) = 0. \tag{1}$$

Hooke's Law in the Cartesian coordinates x_i ($i = 1, 2, 3$) expresses connections between components σ_{ij} of stress tensor and components ε_{ij} of deformation tensor and for the linear elasticity has the form

$$\sigma_{ij} = (\lambda + 2\mu)(\nabla \cdot \mathbf{u})\delta_{ij} + 2\mu\varepsilon_{ij}, \quad \varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right). \quad (i, j = 1, 2, 3) \tag{2}$$

For simplicity we will consider the plane deformation when components of displacement $\mathbf{u}$ have the form $u_x = u(x, y)$, $u_y = v(x, y)$, $u_z = 0$. Let $\Omega \in \mathbb{R}^2$ be the rectangle $\mathbf{r} \in (0, a) \times (0, b)$. The problem to be solved is the next:

$$L\mathbf{u}(\mathbf{r}) = 0, \quad \mathbf{r} \in \Omega; \quad \mathbf{u} = \mathbf{v},\ \mathbf{T}_n = \mathbf{F} \text{ on } y = b; \quad u_x = \sigma_{xy} = 0 \quad \text{on } x = 0 \text{ and } x = a, \tag{3}$$

here $\mathbf{T}_n$ is a stress vector. Thus, we have the Cauchy problem for the Lame equation with the Cauchy data on the upper base and with zero boundary conditions on the lateral sides of rectangle.

3. Method of solution

Let us introduce a function $f_0 = (\lambda + 2\mu)\nabla \cdot \mathbf{u}$. It is known that this function is harmonic. Using boundary conditions from (3) and Hooke's law (2) it can be shown that for this function we have the Cauchy problem. The regularized solution f_0^α of this problem is obtained by using the Liu [3] method

$$\begin{aligned} f_0^\alpha(x,y) &= \sum_{k=1}^{\infty} a_k^\alpha \frac{\sinh[k\pi(b-y)/a]}{\sinh[k\pi b/a]} \sin\frac{k\pi x}{a}, \\ a_k^\alpha &= \frac{2\sinh(k\pi b)/a]}{k\pi + \alpha a \sinh(k\pi b/a)} \int_0^a h(\xi)\sin\frac{k\pi\xi}{a} d\xi, \end{aligned} \tag{4}$$

where α is a parameter of regularization; the function $h(x)$ is expressed in terms of boundary functions from (3). In this method the integral equation of the first kind is solved by Lavrent'ev's regularization method using the Fourier method. The kernel in this integral equation has a termwise separable property. This property is the main reason for a solution to be obtained in a closed form. After finding f_0^α for the components u_x and u_y of displacement we also have the Cauchy problems for the Poisson equations. These problems are solved by the method similar to Liu. Thus, for the solution u_x and u_y of the Cauchy problem (3) we have regularized expressions similar to (4), but with additional regularization parameters β and γ. If the boundary functions in (3) are bounded in closed intervals, then all of these regularized solutions converge uniformly to an exact solution and can be differentiated termwise any number of times. Error estimations for these solutions are obtained. The presented numerical examples show the efficiency of new method.

4. Conclusion

The Liu method for solving the Cauchy problem for the Laplace equation is generalized for the Cauchy problem of the linear elasticity. This allows us to obtain the regularized solution of the problem in the closed form by using the Fourier and Lavrent'ev methods. The obtained regularized solution converges absolutely and uniformly and the error estimation of regularized solution is proved. The numerical examples show the efficiency and robustness of new method. For simplicity we only consider a two–dimensional rectangular domain. The method can be used in the cases of other boundary conditions, it can be generalized for a three–dimensional case.

5. Acknowledgments

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6. References

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