MULTYILAYER PLATE BENDING MODEL
WITH APPLICATION TO A NANO-PLATE BENDING AND FREE VIBRATIONS

N.F.Morozov[1,2]***, P.E.Tovstik***[1,2]***, T.P.Tovstik***[2]
[1]*St.Petersburg State University, St.Petersburg, Russia*
[2]*Institute of Problems of Mechanical Engineering RAS, St. Petersburg, Russia*

1. Introduction

A thin multilayer plate made of linearly elastic homogeneous isotropic layers is studied. The Young moduli ratio of hard and soft layers is supposed to be large. The generalized Timoshenko–Reissner (TR) model is used to describe the bending and the free vibrations of this plate. A rectangular plate with simply supported edges is studied. Then this model is applied to a multilayer graphite nano-plate consisting of graphene layers. The expressions for the bending and for the transversal shear stiffness are proposed. As a result, the explicit formulae for the bending amplitude and for the natural bending frequencies of the nano-plate are delivered.

The 2D model for a transversely isotropic heterogeneous plate of the second-order accuracy with respect to the small thickness parameter $\mu = h/L$ is delivered in [1]. It occurs that compared with the classic Kirchhoff–Love (KL) model, the terms including the transversal shear are the main additional terms. That is why in [2] the generalized (TR) model for a multilayer plate is proposed. This model can be applied to multilayer plates with a great difference in stiffness between the Young moduli of hard and soft layers. This result allows us to apply this model to a graphite plate in which the graphene layers are hard, and Van-der-Waals forces acting between the graphene layers are modeled by soft isotropic layers.

2. Bending deformation and vibrations of a multilayer plate

To describe the bending deformation the equation [2]

$$D\Delta^2 w = F_3 - \frac{D}{\Gamma}\Delta F_3, \quad \Delta(\cdot) = (\cdot)_{,xx} + (\cdot)_{,yy}, \tag{1}$$

based on the TR model for a transversely isotropic homogeneous plate, which is equivalent to a multilayer plate with the layer thicknesses h_k, is used, where

$$\begin{gathered} D = \left\langle (z-a)^2 E_0(z) \right\rangle, \quad a = \frac{\langle z E_0(z)\rangle}{\langle E_0(z)\rangle}, \quad E_0 = \frac{E(z)}{1-\nu^2(z)}, \quad \langle X \rangle = \int_0^h X(z)dz, \\ \frac{1}{\Gamma} = \frac{1}{D^2}\int_0^h \frac{(e(z))^2}{G_{13}(z)}dz, \quad e(z) = \int_0^z (z-a)E_0(z)dz. \end{gathered} \tag{2}$$

Here D is the bending stiffness, a is the neutral layer position, Γ is the transversal shear stiffness, $F_3(x,y)$ is the external transversal force density, $0 \le z \le h = \sum h_k$ is the plate thickness, the functions $E_0(z)$, $G_{13}(z)$ are the piece-wise constant functions in z.

For a rectangular plate $0 \le x \le L_x$, $0 \le y \le L_y$ with the simply supported edges under action of a harmonic external load $F_3(x,y) = F_3^0 \sin r_x x \sin r_y y$, $r_x = \pi x/L_x$, $r_y = \pi x/L_y$, the deflection $w(x,y)$ also is harmonic

$$w(x,y) = w^0 \sin r_x x \sin r_y y, \qquad w^0 = w_b + w_s, \quad w_b = \frac{F_3^0}{Dr^4}, \quad w_s = \frac{F_3^0}{\Gamma r^2}, \quad r^2 = r_x^2 + r_y^2, \tag{3}$$

where w_b and w_s are the bending and the shear parts of deflection, respectively [3].

We rewrite Eq. (3) as $w^0 = w_b(1+g)$, where $g = r^2 D/\Gamma$ is the shear parameter. Taking into account that $D/\Gamma = h^2\xi$, we get $g = \mu^2\xi$, where $\mu = rh$ is the small thickness parameter. If the Young moduli of layers are close to each other, then $\xi \sim 1$ and $g \ll 1$, and the value g may be neglected. In this case the KL model may be used. If $\xi \gg 1$ then the shear deformations become essential and Eq. (3) is to be used while $g \sim 1$. If $g \gg 1$, Eq. (3) is not appropriate.

At the case of free vibrations with frequency ω we take $F_3 = \rho\omega^2 w$, and Eq. (3) gives the approximate expression for the first natural frequency

$$\omega^2 = Dr^4/(\rho(1+g)). \tag{4}$$

Eq. (4) is acceptable also at $g \sim 1$.

3. Model of the multilayer nano-plate bending

Let a graphite plate consist of $n+1$ graphene layers. We model each layer as a thin isotropic plate with the extension stiffness K_0 and with the bending stiffness D_0 [4]. We model the intermediate layers of thicknesses h_0 between graphene layers as isotropic elastic layers with the small stiffness. Then we get a multilayer plate, and we model it as an one-layered homogeneous TR plate (see Section 2). We calculate the equivalent stiffness D and Γ by Eqs. (2).

At an calculation of integrals in Eqs. (2) we neglect the thickness of graphene layers compared with h_0 (then $h = nh_0$), neglect the shear compliance of graphene layers compared with the shear compliance of intermediate layers, and also neglect the extension stiffness of intermediate layers. Then $E_0(z) = K_0 \sum_{k=0}^{n} \delta(z-kh_0)$, where $\delta(z)$ is the Dirac's delta-function. We obtain the equivalent stiffness as:

$$D=(n+1)D_0+\alpha_n h_0^2 K_0, \quad \frac{1}{\Gamma}=\frac{\beta_n h_0^3 K_0^2}{D^2 G_{13}^0}, \quad \alpha_n=\frac{n(n+1)(n+2)}{12}, \; \beta_n=\frac{\alpha_n(n^2+2n+2)}{10}. \tag{5}$$

Here G_{13} is the transversal shear stiffness of the intermediate layers. In the formula for the bending stiffness D the first summand is much larger than the second one and it is to be omitted. Namely, for the multilayer nano-plate we ought to ignore the bending stiffness of separate layers.

Now we may use Eqs. (3) and (4) to calculate the deflection and the frequency of free vibrations of a graphite nano-plate. Numerical examples are given.

4. Conclusions

The formula for the transversal shear stiffness Γ is the main result. The proposed TR model may have a wide field of applications for ordinary multilayer plates. For nano-plates this model has rather a theoretical interest, but its generalization for multilayer carbon nanotubes is promising.

Supported by grants RFBR 16.01.00580, 16.51.52025 MHT and by grant SPbU 6.38.337.2015.

5. References

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