BOUNDARY VALUE PROBLEMS OF STEADY VIBRATIONS IN THE THEORY OF THERMOELASTIC DOUBLE POROSITY MATERIALS

M. Svanadze
Institute for Fundamental and Intedisciplinary Mathematics Research,
Ilia State University, Tbilisi, Georgia

1. Introduction

The materials with double porosity play an important role in many branches of engineering, e.g., the petroleum industry, chemical engineering, geomechanics, and, in recent years, biomechanics. The theories of elasticity and thermoelasticity for materials with double porosity based on the Darcy's law are presented by several authors (see [1], [2] and references therein) and studies in the series of papers [3]-[6]. Recently, on the basis of balance of equilibrated force the theories of elasticity and thermoelasticity for materials with a double porosity structure are presented in [7]. The basic three-dimensional BVPs of the equilibrium theory of elasticity for materials with a double porosity structure are investigated by using the potential method (boundary integral equation method) and the theory of singular integral equations in [8].

The present paper concerns with the linear theory of thermoelasticity for materials with a double porosity structure based on the balance of equilibrated force [7] and the 3D basic boundary value problems (BVPs) of steady vibrations of this theory are investigated. The fundamental solution of the system of equations of steady vibrations is constructed. The representation of general solution for this system is obtained. The Green's formulae and Sommerfeld-Kupradze type radiation conditions are established. The uniqueness theorems for the classical solutions of the BVPs of steady vibrations are proved. The basic properties of surface and volume potentials are established. Finally, on the basis of the potential method and the theory of singular integral equations the existence theorems for the classical solutions of the BVPs of steady vibrations are proved.

2. Basic Equations

Let $\mathbf{x} = (x_1, x_2, x_3)$ be a point of the Euclidean three-dimensional space $\mathbb{R}^3$. In what follows we consider an isotropic and homogeneous elastic material with a double porosity structure that occupies a region of $\mathbb{R}^3$; $\mathbf{u}(\mathbf{x})$ denote the displacement vector, $\mathbf{u} = (u_1, u_2, u_3)$; $\varphi(\mathbf{x})$ and $\psi(\mathbf{x})$ are the changes of volume fractions from the reference configuration corresponding to pores and fissures, respectively; θ is the temperature measured from the constant absolute temperature T_0 $(T_0 > 0)$.

The system of homogeneous equations of steady vibrations in the linear theory of thermoelasticity for materials with a double porosity structure based on the balance of equilibrated force has the following form [7]

$$
\begin{aligned}
&\mu\Delta\mathbf{u} + (\lambda+\mu)\nabla\,\mathrm{div}\mathbf{u} + b\nabla\varphi + d\nabla\psi - \gamma_0\nabla\theta + \rho\omega^2\mathbf{u} = \mathbf{0},\\
&(\alpha\,\Delta + \eta_1)\,\varphi + (\beta\,\Delta - \alpha_3)\,\psi - b\,\mathrm{div}\mathbf{u} + \gamma_1\theta = 0,\\
&(\beta\,\Delta - \alpha_3)\,\varphi + (\gamma\,\Delta + \eta_2)\psi - d\,\mathrm{div}\mathbf{u} + \gamma_2\theta = 0,\\
&k\Delta\theta + i\omega T_0(a\,\theta + \gamma_0\,\mathrm{div}\mathbf{u} + \gamma_1\varphi + \gamma_2\psi) = 0,
\end{aligned}
\tag{1}
$$

where Δ is the Laplacian operator; $\eta_j = \rho_j\omega^2 - \alpha_j$, $j = 1, 2$; ρ is the reference mass density, ρ_1 and ρ_2 are the coefficients of the equilibrated inertia, $\rho_1 > 0$, $\rho_2 > 0$; ω is the angular frequency; $a, b, d, k, \alpha, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma, \lambda$ and μ are constitutive coefficients. We assume that the internal energy of the isotropic and homogeneous elastic materials with a double porosity structure is positive definite.

Let S be a closed smooth surface surrounding the finite domain Ω^+ in $\mathbb{R}^3$, $\bar{\Omega}^+ = \Omega^+ \cup S$, $\Omega^- = \mathbb{R}^3 \setminus \bar{\Omega}^+$, $\mathbf{n}(\mathbf{z})$ is the external unit normal vector to S at $\mathbf{z} \in S$. The basic external BVPs of steady vibrations in the linear theory of thermoelasticity of double-porosity materials are formulated as follows: find a regular (classical) solution $\mathbf{U} = (\mathbf{u}, p_1, p_2, \theta)$ to system (1) in Ω^- satisfying the boundary condition $\lim_{\Omega^- \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{U}(\mathbf{x}) \equiv \{\mathbf{U}(\mathbf{z})\}^- = \mathbf{f}(\mathbf{z})$ in the problem $(I)^-_{\mathbf{f}}$, and $\lim_{\Omega^- \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{R}(\mathbf{D}_\mathbf{x}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{x}) \equiv \{\mathbf{R}(\mathbf{D}_\mathbf{z}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^- = \mathbf{f}(\mathbf{z})$ in the problem $(II)^-_{\mathbf{f}}$, where $\mathbf{f}$ is the known six-component vector function and $\mathbf{R}$ is the stress operator in the considered theory.

3. Basic results

The following results in the considered theory are obtained: the fundamental solution of the system (1) is constructed by elementary functions, the representation of general solution for this system is presented, the Green's formulae and Sommerfeld-Kupradze type radiation conditions are established, the basic properties of surface and volume potentials are obtained, and finally, on the basis of the potential method and the theory of singular integral equations the following uniqueness and existence theorems for the classical solutions of the BVPs of steady vibrations are proved.

Theorem 1. The external BVPs $(I)^-_{\mathbf{f}}$ and $(II)^-_{\mathbf{f}}$ have one regular solution.

Theorem 2. If $S \in C^{2,\nu}$, $\mathbf{f} \in C^{1,\tau}(S)$, $0 < \tau < \nu \leq 1$, then a regular solution of the external BVP $(I)^-_{\mathbf{f}}$ exists, is unique and is represented by sum $\mathbf{U}(\mathbf{x}) = \mathbf{Z}^{(2)}(\mathbf{x}, \mathbf{g}) + (1 - i)\mathbf{Z}^{(1)}(\mathbf{x}, \mathbf{g})$ for $\mathbf{x} \in \Omega^-$, where $\mathbf{Z}^{(1)}(\mathbf{x}, \mathbf{g})$ and $\mathbf{Z}^{(2)}(\mathbf{x}, \mathbf{g})$ are the single-layer and double-layer potentials, respectively, and $\mathbf{g}$ is a solution of the singular integral equation $\frac{1}{2}\mathbf{g}(\mathbf{z}) + \mathbf{Z}^{(2)}(\mathbf{z}, \mathbf{g}) + (1 - i)\mathbf{Z}^{(1)}(\mathbf{z}, \mathbf{g}) = \mathbf{f}(\mathbf{z})$ for $\mathbf{z} \in S$ which is always solvable for an arbitrary vector $\mathbf{f}$.

Theorem 3. If $S \in C^{2,\nu}$, $\mathbf{f} \in C^{0,\tau}(S)$, $0 < \tau < \nu \leq 1$, then a regular solution of the external BVP $(II)^-_{\mathbf{f}}$ exists, is unique and is represented by sum $\mathbf{U}(\mathbf{x}) = \mathbf{Z}^{(1)}(\mathbf{x}, \mathbf{g}) + \mathbf{V}(\mathbf{x})$ for $\mathbf{x} \in \Omega^-$, where $\mathbf{g}$ is a solution of the singular integral equation $-\frac{1}{2}\mathbf{g}(\mathbf{z}) + \mathbf{R}\,\mathbf{Z}^{(1)}(\mathbf{z}, \mathbf{g}) = \mathbf{f}_1(\mathbf{z})$ for $\mathbf{z} \in S$ which is always solvable for an arbitrary vector $\mathbf{f}_1$, $\mathbf{V}$ is a solution of the BVP $(I)^-_{\mathbf{f}_2}$, $\mathbf{f}_1$ and $\mathbf{f}_2$ are the known six-component vector functions.

4. References

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