COMPARISON OF TWO METHODS
FOR NUMERICAL UPSCALING

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1. Introduction

Several approaches to numerical modeling of heterogeneous materials can be found in the literature. Among them, there is a group of discretization-based homogenization methods. They are used in the case when we cannot afford the mesh fine enough to account for the heterogeneities of the material. Instead of that the finest possible mesh is generated and effective stiffness matrices are computed for its elements. This mesh, referring to the macro level, is called the coarse one. Independent fine meshes that appropriately resolve material heterogeneity are generated for every coarse element.

In this paper we present briefly two techniques used for evaluation of the afore-mentioned matrices- local numerical and multiscale (multigrid) homogenization. We enhance them using *hp*-adaptive FEM at both micro and macro level (see e.g. [1, 2]).

2. Local numerical homogenization

Detailed description of local numerical homogenization can be found in [3, 4]. Authors used the method originally for lattice models. However, it is general and subsequently it was used for non-linear continuum mechanics problems [5].

In the further discussion we naturally focus on a single coarse element. Subsequently, we refine it in order to comply with the micro structure. Our problem is to find such a coarse element stiffness matrix that minimizes the squared norm of the difference between fine and coarse mesh solutions. Mathematical formulation is as follows:

Given symmetric fine mesh element stiffness matrices assembled into $\boldsymbol{K_h}$, a non-zero fine mesh load vector $\boldsymbol{f_h}$, interpolation matrix $\boldsymbol{A}$, positive-definite symmetric weight matrix $\boldsymbol{B}$, dimensionless small parameter $\epsilon > 0$, find a symmetric matrix $\boldsymbol{K}_{\boldsymbol{H}}^{\dagger}$ (pseudoinverse of a coarse element stiffness matrix $\boldsymbol{K_H}$) minimizing E, where

$$(1)\qquad E(\boldsymbol{K}_{\boldsymbol{H}}^{\dagger}) = \frac{1}{2}\left\|(\boldsymbol{K}_{\boldsymbol{h}}^{\dagger} - \boldsymbol{A}\boldsymbol{K}_{\boldsymbol{H}}^{\dagger}\boldsymbol{A}^{T})\boldsymbol{f_h}\right\|_{\boldsymbol{B}}^{2} + \frac{\epsilon}{2}\left\|\boldsymbol{K}_{\boldsymbol{h}}^{\dagger} - \boldsymbol{A}\boldsymbol{K}_{\boldsymbol{H}}^{\dagger}\boldsymbol{A}^{T}\right\|_{F,\boldsymbol{B}}^{2}\left\|\boldsymbol{f_h}\right\|_{2}^{2}$$

and $\|\boldsymbol{x}\|_2 = \sqrt{trace(\boldsymbol{x}^T\boldsymbol{x})}$ denotes Euclidean norm, $\|\boldsymbol{x}\|_{\boldsymbol{B}} = \sqrt{trace(\boldsymbol{x}^T\boldsymbol{B}\boldsymbol{x})}$ denotes Euclidean norm weighted with $\boldsymbol{B}$, $\|\boldsymbol{X}\|_{F,\boldsymbol{B}} = \sqrt{trace(\boldsymbol{X}^T\boldsymbol{B}\boldsymbol{X})}$ denotes Frobenius norm weighted with $\boldsymbol{B}$.
$\boldsymbol{x}$ and $\boldsymbol{X}$ are arbitrary column vector and arbitrary matrix, respectively.

3. Multiscale homogenization

Multiscale FEM (MsFEM) [7] is equivalent to multigrid homogenization [6, 8]. Inter-grid operators between coarse and fine mesh shape functions can be found by solving the problem formulated in a weak form as:
Find $\boldsymbol{\Phi}(\boldsymbol{x}) \in V_0 + \hat{\boldsymbol{\Phi}}$ such that

$$(2)\qquad \int_{\Omega} \boldsymbol{\sigma}(\boldsymbol{\Phi}) : \boldsymbol{\varepsilon}(\boldsymbol{v})d\Omega = \int_{\Omega} \boldsymbol{v} \cdot Reg\left[div\boldsymbol{\sigma}(\boldsymbol{\Psi})\right] d\Omega \quad \forall \boldsymbol{v} \in V_0$$

where $\boldsymbol{\psi}$ is a coarse mesh vector valued shape function, $\boldsymbol{\Phi}$ is its interpolant and Reg denotes regular part of the derivative. Ω stands for the coarse element domain.

Solution to 2 obtained for every vector valued shape function $\boldsymbol{\psi}$ constitutes the interpolation operator $\boldsymbol{I}_{M\times N}$, where M is the number of macro element degrees of freedom and N denotes the number of fine mesh degrees of freedom.

Assuming that restriction operator $\boldsymbol{R} = \boldsymbol{I}^T$ one computes $\boldsymbol{K}_H = \boldsymbol{I}^T \boldsymbol{K}_h \boldsymbol{I}$ and $\boldsymbol{f}_H = \boldsymbol{I}^T \boldsymbol{f}_h$, where $\boldsymbol{K}_H$ denotes effective coarse element stiffness matrix and $\boldsymbol{f}_H$ is coarse element load vector. $\boldsymbol{K}_h$ and $\boldsymbol{f}_h$ stand for fine mesh stiffness matrix and load vector, respectively.

4. Final remarks

General algorithms of the afore presented methods are very similar. The only difference is the way of effective stiffness matrices computation. Thorough comparison of both methods will be presented during the conference. Numerical results concerning Fichera corner with various distribution of inclusions will be shown. Large reduction of degrees of freedom was obtained without reasonable additional modeling error. The preliminary tests indicate that efficiency of the MsFEM is superior in comparison with the other approach.

Selected numerical implementation aspects and future plans concerning both homogenization methods will be also presented during the conference.

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