ON THERMODYNAMICALLY CONSISTENT FORM OF NONLINEAR EQUATIONS OF THE COSSERAT THEORY

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1. Introduction

The main goal of this talk is to reduce the complete system of equations for the dynamics of a medium with couple stresses under finite strains and particle rotations to a thermodynamically consistent system of laws of conservation. Such system enables analyzing generalized solutions with surfaces of discontinuity of stresses and velocities, and allows to obtain the integral estimates that guarantee the uniqueness and continuous dependence on the initial data of solutions of the Cauchy problems and the boundary-value problems with dissipative boundary conditions.

For this the special natural measure of curvature is chosen that is a strain state characteristic independent of deformation method. The nonlinear constitutive equations of the couple stress theory are constructed using the method of internal thermodynamic parameters of state. The linearization of these equations in isotropic case yields the Cosserat continuum equations, where material resistance to the change in curvature is characterized by a single coefficient as against the three independent coefficients of the classical theory, [1]. So, it turns out that the developed variant of the model gives an adequate description of generalized plane stress state in an isotropic micropolar medium while the classical one describes this state exclusively at a certain ratio of elastic coefficients characterizing resistance of a material to the change in internal curvature.

2. Special tensor of curvature

The translational motion of a particle in a medium possessing microstructure is described by an equation $x = \xi + u$, connecting the Lagrangian ξ and Eulerian x vectors of centers of masses with the displacement vector $u(\xi, t)$. The independent rotation of a particle is defined by an orthogonal rotation tensor $R(\xi, t)$. The antisymmetric tensor of angular velocity of a particle is calculated by the formula: $\Omega = \dot{R} \cdot R^*$ (hereinafter star denotes the conjugate). As a measure of deformation of an infinitely small element, it is assumed to take the tensor $\Lambda = R^* \cdot x_\xi$. By differentiating with respect to time, it is found that the latter tensor satisfies the equation:

$$(1) \qquad R \cdot \dot{\Lambda} = v_\xi - \Omega \cdot x_\xi,$$

where $v = \dot{x}$ is the vector of velocity of translational motion. Aside from the tensor Λ, a special curvature tensor M is used, calculated in terms of the rotation tensor R and its derivatives with respect to the Lagrangian variables in the Cartesian coordinate system $R_{,k} = \partial R / \partial \xi_k$ $(k = 1, 2, 3)$. Let $M^{(k)} = R_{,k} \cdot R^*$ be the antisymmetric curvature tensors along the coordinate lines. The Darboux vectors fitting with these tensors are assigned by the columns of M. Differentiating $M^{(k)}$ with respect to time and Ω with respect to the variables ξ_k yields kinematic equations that admit the tensor representation:

$$(2) \qquad \dot{M} = \omega_\xi + \Omega \cdot M.$$

Note, that it differs from the equation for commonly used curvature measures, [2, 3]. It follows from (2) that M is neither an invariant nor an indifferent tensor, i.e. it changes both under rotation of the current configuration and under rotation of the original configuration. By the same law goes the transformation of the distortion tensor x_ξ, for instance, which is used to determine the invariant strain measure $x_\xi^* \cdot x_\xi$, involved in the Lagrangian representation of motion in a classic elastic medium, and an indifferent measure $x_\xi \cdot x_\xi^*$, included in the Eulerian representation. The both measures are independent of rotation of a medium element as a rigid whole. Similarly, the invariance is the property of the product $M^* \cdot M$, that must be used as an independent parameter of state to construct constitutive equations accounting for the couple properties of a medium, and that leads to a thermodynamically consistent system of conservation laws.

3. Thermodynamically consistent system

The system of equations of the dynamics of a medium with couple stresses is constructed based on the integral laws of impulse, momentum and energy conservation. The principles of thermodynamics goes into the constitutive equations for stress tensor σ and couple stress tensor m:

$$R^* \cdot \sigma = \frac{\partial \Phi}{\partial \Lambda}, \quad m = \frac{\partial \Phi}{\partial M}, \tag{3}$$

and a supplementary equation $m^* : (\Omega \cdot M) = 0$, which is satisfied automatically because the stress potential Φ depends not of M, but of the symmetric tensor $M^* \cdot M$.

The equations (1) – (3) allows representing the model by a thermodynamically consistent system in the following sense: it is possible to indicate generating potentials L^0 and L^j, the use of which modifies the complete system of equations to the next form:

$$\frac{\partial}{\partial t} \frac{\partial L^0(D\,U)}{\partial U} = \frac{\partial}{\partial \xi_j} \frac{\partial L^j(U)}{\partial U} + F(D,U), \quad \frac{\partial D}{\partial t} = G(D,U). \tag{4}$$

Here U is the column-vector composed of unknown functions, namely, projections of vectors of velocity of translational motion and angular velocity, components of tensors of stresses and couple stresses; D is the nonsingular matrix, the non-zero and non-unit coefficients of which are the components of tensor of rotation R; F and G are the preset vector and matrix functions.

The system (4) can be written as symmetric t-hyperbolic system which is investigated by means of well-designed methods used, e.g. in [4].

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5. References

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