ON NON-LOCAL MATERIALS, INTERNAL LENGTH AND FRACTIONAL CALCULUS

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1. Non-local materials

Non-local materials were already studied in the 1960s by several authors (for example [1]) as a part of continuun mechanics. When material instability gained more interest, non-local behaviour appeared again [2], because instability zones exhibited singular properties for local constitutive equations. Such works used the gradient of strain tensors to include non-locality into the constitutive equation

(1) $$F(\sigma, \varepsilon, \nabla\varepsilon, \nabla^2\varepsilon, \ldots) = 0.$$

2. Gradient materials and internal length

Most gradient theories concentrate on the second gradient, let the constitutive equation be in rate form

(2) $$\dot{\sigma} = \tilde{c}_1\dot{\varepsilon} + \tilde{c}_2\ddot{\varepsilon} - \tilde{c}_3\frac{\partial^2\dot{\varepsilon}}{\partial x^2}$$

then the set of the basic equations of continua consists of (2) and the equation of motion together with the kinematic equation

$$\rho\dot{v} = \frac{\partial\sigma}{\partial x}, \quad \dot{\varepsilon} = \frac{\partial v}{\partial x}.$$

By transforming them into the velocity field and using new variables

$$y_1 = v, y_2 = \dot{v}$$

a dynamical system

(3) $$\dot{y}_1 = y_2,$$

(4) $$\dot{y}_2 = \left(c_1\frac{\partial^2}{\partial x^2} - c_3\frac{\partial^4}{\partial x^4}\right)\dot{y}_1 + c_2\frac{\partial^2}{\partial x^2}y_2,$$

is obtained, where

$$c_i = \frac{\tilde{c}_i}{\rho}, \quad (i = 1, 2, 3).$$

Its characterisic equation for λ reads

(5) $$\lambda^2 y_1 - \lambda c_2\frac{\partial^2}{\partial x^2}y_1 - \left(c_1\frac{\partial^2}{\partial x^2} - c_3\frac{\partial^4}{\partial x^4}\right)y_1 = 0.$$

The critical eigenfunction of (5) at the loss of stability ($c_1 = c_{1crit} < 0$) is

(6) $$y_1 = \exp\left(ix\sqrt{-\frac{c_{1crit}}{c_3}}\right)$$

and

$$\ell^* := \pi\sqrt{-\frac{c_3}{c_{1crit}}} \tag{7}$$

can be identified as internal length.

However, by comparing (1) and (2) several question arise: is there a Taylor expansion for ε? Why the first order gradient is missing? Anyway, for the basic eqations, even at the simplest $\tilde{c}_2 = \tilde{c}_3 = 0$ constitutive equation, there is a gradient dependent term

$$\dot{\sigma} = \tilde{c}_1 \frac{\partial v}{\partial x}. \tag{8}$$

3. Fractional calculus

Following the idea of [3] (8) can be generalized to fractional derivatives

$$\dot{\sigma} = \tilde{c}_1 \frac{1}{2}\left({}^C D^{\alpha}_{a+} u(x) - {}^C D^{\alpha}_{L-} u(x)\right), \tag{9}$$

where ${}^C D^{\alpha}_{a+} u(x)$ and ${}^C D^{\alpha}_{L-} u(x)$ are α-th fractional derivatives with respect to x for a rod of lenght $L - a$, thus by evaluating them

$$\begin{gathered}({}^C D^{\alpha}_{a+} u(x) - {}^C D^{\alpha}_{L-} u(x)) = \\ \frac{1}{2\Gamma(1-\alpha)}\left(\int_a^x (x-\xi)^{-\alpha}\frac{\partial u(\xi,t)}{\partial \xi} d\xi + \int_x^L (-x+\xi)^{-\alpha}\frac{\partial u(\xi,t)}{\partial \xi} d\xi\right)\end{gathered}$$

Here (and consequently in (9)) non-locality is present as forward and backward integrals along the rod.

Moreover, the inclusion of non-locality by using fractional calculus solves an other problem of conventional gradient theories. In dynamic problems (and stability is always a dynamic problem) the existence of wave solution is required for the basic equations. Such condition excludes several forms for constitutive equations [4], including that one when the only terms with second derivative is ε_{xx}.

4. References

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