IMPLICIT CONSTITUTIVE RELATIONS FOR THERMOELASTIC BODIES

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1. Introduction

Recently some new constitutive relations have been proposed to study the behaviour of elastic bodies [1, 2]. One of such constitutive relations is of the form [2]

(1) $\mathfrak{F}(\mathbf{T},\mathbf{B})=\mathbf{0}$,

where $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{B}=\mathbf{F}\mathbf{F}^{\mathrm{T}}$ is the left Cauchy-Green strain tensor and $\mathbf{F}$ is the deformation gradient. In general from (1) it is not possible to express the stresses as functions of the strains and vice-versa. These kind of constitutive relations cannot be classified as either Cauchy or Green elastic bodies. An interesting subclass of (1) can be found assuming that $|\nabla\mathbf{u}|\sim O(\delta)$, where $\delta\ll 1$ and where $\mathbf{u}$ is the displacement field. In such a case since $\mathbf{B}\approx\mathbf{I}+2\boldsymbol{\varepsilon}$ (where $\boldsymbol{\varepsilon}=\frac{1}{2}(\nabla\mathbf{u}+\nabla\mathbf{u}^{\mathrm{T}})$ is the linearized strain tensor) from (1) we obtain

(2) $\boldsymbol{\varepsilon}=\mathfrak{f}(\mathbf{T})$.

The relations (1) and (2) could have many potential uses in rock mechanics, fracture mechanics and biomechanics [3, 4].

In the present work we study a possible extension of the above implicit constitutive theories for the case of modelling the behaviour of a thermo-elastic body.

2. Implicit constitutive relations

In this work we use the second Piola-Kirchhoff stress tensor $\mathbf{S}$ and the Lagrange strain tensor $\mathbf{E}$, which are defined as $\mathbf{S}=J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-\mathrm{T}}$, $\mathbf{E}=\frac{1}{2}(\mathbf{F}^{\mathrm{T}}\mathbf{F}-\mathbf{I})$, in order to propose implicit constitutive relations. We consider the following implicit relation for a thermo-elastic body

(3) $\mathfrak{G}(\mathbf{S},\mathbf{E},\theta)=\mathbf{0}$,

where θ is the temperature of the body. In the case $\mathfrak{G}$ is an isotropic relation (3) becomes

(4)
$$\alpha_0\mathbf{I}+\alpha_1\mathbf{S}+\alpha_2\mathbf{S}^2+\alpha_3\mathbf{E}+\alpha_4\mathbf{E}^2+\alpha_5(\mathbf{E}\mathbf{S}+\mathbf{S}\mathbf{E})+\alpha_6(\mathbf{E}^2\mathbf{S}+\mathbf{S}\mathbf{E}^2)+\alpha_7(\mathbf{S}^2\mathbf{E}+\mathbf{E}\mathbf{S}^2)+\alpha_8(\mathbf{S}^2\mathbf{E}^2+\mathbf{E}^2\mathbf{S}^2)=\mathbf{0},$$

where α_i, $i=0,1,2,...,8$ are scalar functions that depend on θ and the invariants defined from $\mathbf{S}$ and $\mathbf{E}$. In the special case that $|\nabla\mathbf{u}|\sim O(\delta)$, $\delta\ll 1$ we have $\mathbf{E}\approx\boldsymbol{\varepsilon}$ and $\mathbf{S}\approx\mathbf{T}$ and from (4) we obtain a constitutive equation of the form

(5) $\boldsymbol{\varepsilon}=\beta_0\mathbf{I}+\beta_1\mathbf{T}+\beta_2\mathbf{T}^2$,

where β_j, $j=0,1,2$ are scalar functions that depend on θ and the three independent invariants of $\mathbf{T}$.

An additional relation is needed, which is the vector implicit relation

(6) $\mathfrak{h}(\mathbf{T},\theta,\nabla\theta,\mathbf{q},\dot{\mathbf{q}})=\mathbf{0}$,

where $\mathbf{q}$ is the heat flux. In the case $\mathfrak{h}$ is an isotropic vector function we have (see [6])

(7) $\xi_0\nabla\theta+\xi_1\mathbf{T}\nabla\theta+\xi_2\mathbf{T}^2\nabla\theta+\xi_3\mathbf{q}+\xi_4\mathbf{T}\mathbf{q}+\xi_5\mathbf{T}^2\mathbf{q}+\xi_6\dot{\mathbf{q}}+\xi_7\mathbf{T}\dot{\mathbf{q}}+\xi_8\mathbf{T}^2\dot{\mathbf{q}}=\mathbf{0}$,

where the scalar functions ξ_i, $i=0,1,...,8$ depend on θ and the invariants formed with $\mathbf{T}$, $\nabla\theta$, $\mathbf{q}$ and $\dot{\mathbf{q}}$. This relation would be a generalization of the Fourier's model for heat transfer presented, for example, in [5].

3. Incompressibility constraint

If $|\nabla \mathbf{u}| \sim O(\delta)$, $\delta \ll 1$ the incompressibility constraint is $\mathrm{tr}\boldsymbol{\varepsilon} = g(\theta)$, where we have assumed that the volume of the body can be affected by changes in the temperature, which is represented by the function $g(\theta)$. If we further assume that there exists a scalar function $\Pi = \Pi(I_1, I_2, I_3, \theta)$, where $I_1 = \mathrm{tr}\mathbf{T}$, $I_2 = \frac{1}{2}\mathrm{tr}(\mathbf{T}^2)$, $I_3 = \frac{1}{3}\mathrm{tr}(\mathbf{T}^3)$, and where $\boldsymbol{\varepsilon} = \frac{\partial \Pi}{\partial \mathbf{T}}$, we have $\beta_{i-1} = \frac{\partial \Pi}{\partial I_i}$, $i = 1, 2, 3$, and the constraint becomes the first order partial differential equation for Π

$$(8) \quad 3\frac{\partial \Pi}{\partial I_1} + I_1 \frac{\partial \Pi}{\partial I_2} + 2I_2 \frac{\partial \Pi}{\partial I_3} = g(\theta).$$

The solution of this equation is of the form (see [6]) $\Pi(I_1, I_2, I_3, \theta) = g(\theta)\frac{I_1}{3} + \bar{\Pi}(\bar{I}_1, \bar{I}_2, \theta)$, where we have defined $\bar{I}_1 = I_1 - \frac{I_1^2}{6}$, and $\bar{I}_2 = I_3 - \frac{2}{3}I_1 I_2 + \frac{2}{27}I_1^3$.

4. Boundary value problem

The displacement field $\mathbf{u}$, the Cauchy stress tensor $\mathbf{T}$, the temperature θ and the heat flux $\mathbf{q}$ (where in total we have 13 components) have to satisfy the equation of motion

$$(9) \quad \mathrm{div}\mathbf{T} + \rho \mathbf{b} = \rho \ddot{\mathbf{u}},$$

the constitutive relations 5 and 7 and the first law of thermodynamics

$$(10) \quad \rho \dot{\epsilon} = \mathrm{tr}(\mathbf{T}\dot{\boldsymbol{\varepsilon}}) + \mathrm{div}\mathbf{q} + \rho r,$$

where ϵ is the internal energy of the body and r is the rate of heat per unit of mass generated by the body. In total we would have 13 equations.

5. Further remarks

Regarding the second law of thermodynamics, in [6] there will be a study about how to impose restrictions on these constitutive relations such that such law is obeyed. In the same communication several boundary value problems will be solved, to see the potential uses of the theories presented in this work.

6. References

[1] Rajagopal, K.R.: On implicit constitutive theories. Appl. Math. 48:279–319, 2003.

[2] Rajagopal, K.R.: The elasticity of elasticity. Z. Angew. Math. Phys. 58:309–317, 2007.

[3] Johnson, P.A., Rasolfosaon, P.N.J.: Manifestation of nonlinear elasticity in rock: Convincing evidence over large frequency and strain intervals from laboratory studies. Nonlinear Proc. Geoph. 3:77–88, 1996.

[4] Freed, A.D.: Soft Solids. A Primer to the Theoretical Mechanics of Materials. Birkhäuser 2014.

[5] Ignaczak, J., Martin Ostoja-Starzewski, M.: Thermoelasticity with Finite Wave Speeds, Oxford University Press 2009.

[6] Bustamante, R., Rajagopal, K.R.: On implicit constitutive relations for thermo-elastic bodies. (In preparation), 2016.