MATERIAL SPIN AND FINITE HYPO-ELASTICITY FOR TWO-DIMENSIONAL ORTHOTROPIC MEDIA

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Introduction

The problem of motion decomposition onto rigid and deformable parts is well known in the nonlinear continuum mechanics of solids from the earlier works of S.Zaremba and G.Jaumann. This problem partially coincides with defining the indifferent corotational rate. Recognizing rigid motion of the material within finite deformations is the only way to observe strain history and write elastic law with the changeless elastic moduli. The measures of rotation often being exploiting such as orthogonal tensor $\mathbf{R}$ from the polar decomposition $\mathbf{F}=\mathbf{R}\cdot\mathbf{U}=\mathbf{V}\cdot\mathbf{R}$ of the deformational gradient $\mathbf{F}$ (operation $\cdot$ defines single scalar production) and its spin tensor $\mathbf{\Omega_R}=\dot{\mathbf{R}}\cdot\mathbf{R}^{\mathrm{T}}$ or the vorticity tensor $\mathbf{W}$, or the logarithmic spin tensor $\mathbf{\Omega}_{\log}$ (recently introduced by A.Meyers, H.Xiao, O.Bruhns) have no relation to the real material frame rotation. The latter is supposed to be associated only with the material symmetry axes evolution and has to be defined in a special manner. The approach of recognizing the material rotation considered below requires preserving the type of material symmetry under deformation. All the monocrystals being deformed without phase transitions and some composites are appropriate instances. For the simplicity of the material spin introduction a two-dimensional orthotropic medium and two-dimensional deformations are further considered.

1. Material Rotation and Material Spin

Definition: the material symmetry rotation tensor $\mathbf{Q}^{\#}$ in any chosen material point is a proper orthogonal second rank tensor connecting the actual, but not deformed material symmetry axes $\mathbf{a}_i^{\#}$ (obtained in the actual configuration by releasing a small material neighborhood of the point considered) with the initial ones $\mathbf{a}_i$: $\mathbf{a}_i^{\#}=\mathbf{Q}^{\#}\cdot\mathbf{a}_i$, $\mathbf{Q}^{\#}=\mathbf{a}_i^{\#}\mathbf{a}^i$, $\mathbf{a}_i\cdot\mathbf{a}^j=\delta_i^j$.

The deformed material axes in the actual configuration are $\hat{\mathbf{a}}_i=\mathbf{F}\cdot\mathbf{a}_i$. Suppose that in the releasing process the material volume reaches its final reloaded state corresponding to the minimal summary path of all material points (across the immovable mass centre of the volume) from the deformed configuration into the one released. In the considered case the unit rotation axis $\mathbf{n}$ is fixed and orthogonal to the plane of vectors $\mathbf{a}_i$, $|\mathbf{a}_1|\neq|\mathbf{a}_2|$, $\mathbf{a}_1\perp\mathbf{a}_2$. Then the following equation is valid:

$$(1)\qquad d\left[(\mathbf{a}_1\mathbf{a}_1+\mathbf{a}_2\mathbf{a}_2):(\mathbf{F}-\mathbf{Q}^{\#}(s))^{\mathrm{T}}\cdot(\mathbf{F}-\mathbf{Q}^{\#}(s))\right]/\,ds=0\,.$$

So, the only rotation angle s has to be found. The solution of problem (1) takes the form

$$(2)\qquad \mathrm{tg}(s)=(\kappa\,\mathbf{a}_1\mathbf{a}_2-\kappa^{-1}\mathbf{a}_2\mathbf{a}_1):\mathbf{F}\big/(\mathbf{a}_1\mathbf{a}_1+\mathbf{a}_2\mathbf{a}_2):\mathbf{F}\,,\ \text{where}\ \kappa=|\mathbf{a}_1|/|\mathbf{a}_2|\,.$$

The corresponding material spin is equal to $\mathbf{\Omega}^{\#}=\dot{s}\,\mathbf{n}\times\mathbf{I}$, where $\mathbf{I}$ is unity, $\times$ is vector product. Deforming by pure stretching and two types of simple shear in the material axes basis in the form of

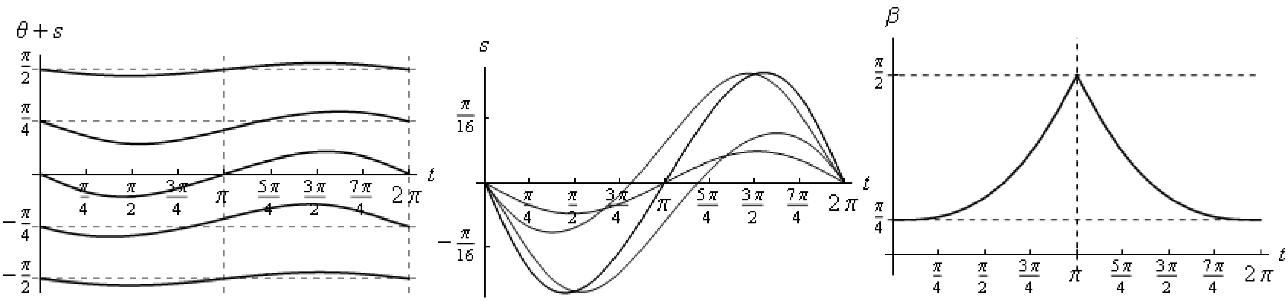
$$\mathbf{F}_{(1)}=\lambda_1\mathbf{a}_1\mathbf{a}^1+\lambda_2\mathbf{a}_2\mathbf{a}^2\,,\ \mathbf{F}_{(2)}=\gamma\,\mathbf{a}_1\mathbf{a}_2/|\mathbf{a}_1||\mathbf{a}_2|+\mathbf{a}_1\mathbf{a}^1+\mathbf{a}_2\mathbf{a}^2\,,\ \mathbf{F}_{(3)}=\gamma\,\mathbf{a}_2\mathbf{a}_1/|\mathbf{a}_1||\mathbf{a}_2|+\mathbf{a}_1\mathbf{a}^1+\mathbf{a}_2\mathbf{a}^2$$

with $\mathbf{a}^i=\mathbf{a}_i/|\mathbf{a}_i|^2$, leads to the corresponding material rotations in the following fashions:

$$\mathrm{tg}(s_{(1)})=0\,,\ \mathrm{tg}(s_{(2)})=-\gamma/(1+\kappa^2)\,,\ \mathrm{tg}(s_{(3)})=\gamma/(1+\kappa^{-2})\,.$$

For the case of small displacements gradient $\mathbf{F} \approx \mathbf{I} + \boldsymbol{\varepsilon} - \boldsymbol{\omega}$, where $\boldsymbol{\varepsilon}$ is a linear small deformation tensor and $\boldsymbol{\omega}$ is a small rotation tensor, the small *material rotation* will be $\mathrm{tg}(s) \approx \mathbf{a}_1 \cdot \boldsymbol{\omega} \cdot \mathbf{a}_2 / |\mathbf{a}_1||\mathbf{a}_2|$.

If initial orientation of the material axes is posed by angle θ of the axis $\mathbf{a}_1$ with respect to the laboratory frame, then expression (2) gives the material axes orientation at an arbitrary instance of time. Its evolution under two-dimensional homogeneous deformation along the elliptic closed path $x^1 = X^1 + aH^{-1}\sin(t)X^2$, $x^2 = X^2 + bH^{-1}(1-\cos(t))X^2$ (with Eulerian laboratory coordinates x^i, material Lagrangian coordinates X^i, elliptic half-axes *a* and *b*, initial height of the specimen *H*) is shown for different initial angles θ in Fig.1, (a). The axes rotation function is smooth and evidently depends on the initial orientation θ in contrast with the orientation β of tensor **V** eigenvectors.



(a) (b) (c)

Fig.1. Time dependences of the material symmetry axes orientations (a), their rotations (b) and orientation of the first eigenvector of tensor **V** (c); $H = 1$, $a = 0.5$, $b = 0.25$, $\kappa = 0.5$

2. Hypo-elastic Anisotropic Law

The problem of postulating the rate form of elasticity for anisotropic materials was discussed by C.A.Truesdell in early 1950s. A widely used hypo-elastic law with Zaremba-Jaumann derivative is suitable for isotropic materials only. The above introduced material rotation allows suggesting hypo-elastic anisotropic law based on the material spin. The J.Rychlewski's theory of invariant representation of tensor functions continued by K.Kowalczyk-Gajewska is pretty convenient for this purpose. For the medium being considered a linear tensor of elastic properties is taking the form $\mathbf{C} = \lambda_{\mathrm{I}}\boldsymbol{\omega}_{\mathrm{I}}\boldsymbol{\omega}_{\mathrm{I}} + \lambda_{\mathrm{II}}\boldsymbol{\omega}_{\mathrm{II}}\boldsymbol{\omega}_{\mathrm{II}} + \lambda_{\mathrm{III}}\boldsymbol{\omega}_{\mathrm{III}}\boldsymbol{\omega}_{\mathrm{III}}$, where λ_K are Kelvin moduli, $\boldsymbol{\omega}_{\mathrm{I}} = |\mathbf{a}_1|^{-2}\,\mathbf{a}_1^{\#}\mathbf{a}_1^{\#}$, $\boldsymbol{\omega}_{\mathrm{II}} = |\mathbf{a}_2|^{-2}\,\mathbf{a}_2^{\#}\mathbf{a}_2^{\#}$, $\boldsymbol{\omega}_{\mathrm{III}} = |\mathbf{a}_1|^{-1}|\mathbf{a}_2|^{-1}(\mathbf{a}_1^{\#}\mathbf{a}_2^{\#} + \mathbf{a}_2^{\#}\mathbf{a}_1^{\#})/\sqrt{2}$. The forth elastic parameter of a two-dimensional orthotropic material is the angle $\theta + s(t)$ involved into tensors $\boldsymbol{\omega}_K$. Material derivative of tensor $\boldsymbol{\omega}_K$ is equal to $\dot{\boldsymbol{\omega}}_K = \boldsymbol{\Omega}^{\#} \cdot \boldsymbol{\omega}_K - \boldsymbol{\omega}_K \cdot \boldsymbol{\Omega}^{\#}$, so the finite form of the linear elastic law $\mathbf{T} = \mathbf{C} : \mathbf{E}$ (where **T** and **E** are certain stress and strain tensors) transforms into the rate form $\mathbf{T}^{\boldsymbol{\Omega}^{\#}} = \mathbf{C} : \mathbf{E}^{\boldsymbol{\Omega}^{\#}}$ due to the fact that all invariant moduli λ_K are constant in the rotating basis $\mathbf{a}_i^{\#}$. The above corotational derivative is defined as $\mathbf{A}^{\boldsymbol{\Omega}^{\#}} = \dot{\mathbf{A}} + \mathbf{A} \cdot \boldsymbol{\Omega}^{\#} - \boldsymbol{\Omega}^{\#} \cdot \mathbf{A}$. The implied early decomposition of motion $\mathbf{F} = \mathbf{F}_{def} \cdot \mathbf{Q}^{\#}$ onto deformable $\mathbf{F}_{def} = \mathbf{R}_{def} \cdot \mathbf{U}_{def} = \mathbf{V}_{def} \cdot \mathbf{R}_{def}$ and rigid parts results into $\mathbf{V}_{def} = \mathbf{V}$, $\mathbf{U}_{def} = \mathbf{Q}^{\#} \cdot \mathbf{U} \cdot \mathbf{Q}^{\#\mathrm{T}}$. Thus, the strain measures of Seth-Hill family including logarithmic strain tensor may be written in the terms of material symmetry axes and used with the introduced hypo-elastic law. Stress and elastic energy evolutions along several complicated closed deformational paths are studied in the work that aims at demonstrating some benefits of the suggested anisotropic hypo-elastic law.

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